Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 2. pp. 171 – 183
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.2.171

CONGRUENCES MODULO 7 AND 11 FOR (s,t)-REGULAR BIPARTITIONS

RANGANATHA D

Abstract. Let $B_{s,t}(n)$ denote the number of (s,t)-regular bipartitions of n. In this paper, we prove several infinite families of congruences modulo s for $B_{s,t}(n)$, where $(s,t) \in \{(7,5),(7,25),(7,125),(11,5)\}$.

2010 Mathematics Subject Classification. 05A15, 05A17, 11P83.

Keywords and phrases. Congruences, Bipartitions, Regular partitions.

1. Introduction

Throughout the paper, we will use the following notation:

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),\ |q|<1.$$

By the binomial theorem, it is easy to check that, for any prime p,

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}. \tag{1}$$

Let k be a positive integer. A k-regular partition is a partition where none of its parts are divisible by k. Let $b_k(0)=1$ and for $n \geq 1$, let $b_k(n)$ denote the number of k-regular partitions of n. The generating function for $b_k(n)$ is given by

$$\sum_{n=0}^{\infty} b_k(n)q^n = \frac{(q^k;q^k)_\infty}{(q;q)_\infty}.$$

Many results on the arithmetic of $b_k(n)$ modulo m have been proven for various values of k and m (see, for example, [3, 11, 15]).

A bipartition (λ,μ) of n is a pair of partitions (λ,μ) such that the sum of all the parts of λ and μ equal n. For positive integers $s \geq 2$ and $t \geq 2$, an (s,t)-regular bipartition of n is a bipartition (λ,μ) of n such that λ is a s-regular partition and μ is a t-regular partition. For example, let $\lambda_1 = 6+3+1+1$ and $\mu_1 = 5+4+2$. Then (λ_1,μ_1) is a $(5,7)$-regular partition of 22. Let $B_{s,t}(n)$ denote the number of (s,t)-regular bipartitions of n. The generating function for $B_{s,t}(n)$ is given by

$$\sum_{n=0}^{\infty} B_{s,t}(n)q^n = \frac{(q^s;q^s)_\infty(q^t;q^t)_\infty}{(q;q)_\infty^2}. \tag{2}$$

Motivated by the works of Ramanujan on congruences for unrestricted partition function $p(n)$, many researchers considered the function $B_{s,t}(n)$ and studied its congruence properties. For example, Lin [6, 7] discovered some

The author would like to thank Prof. Chadrashekar Adiga for his guidance.

infinite families of congruences modulo 3 for $B_{7,7}(n)$ and $B_{13,13}(n)$, namely, for $\alpha \geq 2$ and $n \geq 0$,

$$B_{7,7}\Big(3^\alpha n + \frac{5 \cdot 3^{\alpha-1} - 1}{2}\Big) \equiv \quad \pmod 3$$

and

$$B_{13,13}\Big(3^\alpha n + 2 \cdot 3^{\alpha-1} - 1\Big) \equiv 0 \pmod 3.$$

Dou [5] established an infinite family of congruences modulo 11 for $B_{3,11}(n)$ by using relations between certain cubic theta functions. Xia and Yao [16] proved some infinite families of congruences modulo 3, 5 and 7 for $B_{3,s}(n)$, $B_{5,s}(n)$ and $B_{3,7}(n)$, respectively. Soon after, Wang [14] established several congruences modulo powers of 5 for $B_{5,5}(n)$. Very recently, Adiga et al. [1, 2] discovered several infinite families of congruences modulo s for $B_{s,t}(n)$ where $(s,t) \in \{(3,7),(7,2),(7,5^\beta),(11,3^\beta),(11,5^\beta)\}$ for large values of β. For example, they proved the following theorems:

Theorem 1.1. *For all integers $n, \beta, \alpha \geq 0$ with $\beta \geq 16(\alpha+1)$, we have*

$$B_{7,5^\beta}\Big(5^{16\alpha+15}(5n+j) + 5 \cdot \frac{5^{16\alpha+14} - 1}{24}\Big) \equiv 0 \pmod 7,$$

where $j \in \{0,2,3,4\}$.

Theorem 1.2. *For all integers $n, \beta, \alpha \geq 0$ with $\beta \geq 22\alpha + 17$, we have*

$$B_{11,5^\beta}\Big(5^{22\alpha+17} n + 3 \cdot \frac{5^{22\alpha+16} - 1}{8}\Big) \equiv 0 \pmod{11}.$$

In this paper, we find numerous congruences modulo s for $B_{s,t}(n)$, where $(s,t) \in \{(7,5),(7,25),(7,125),(11,5)\}$. The main results of this paper can stated as follows:

Theorem 1.3. *For all integers $n, \alpha \geq 0$, we have*

$$B_{7,5}\Big(5^{4\alpha+2} n + \frac{13 \cdot 5^{4\alpha+1} - 5}{12}\Big) \equiv 0 \pmod 7, \tag{3}$$

$$B_{7,5}\Big(5^{4\alpha+2} n + \frac{37 \cdot 5^{4\alpha+1} - 5}{12}\Big) \equiv 0 \pmod 7, \tag{4}$$

$$B_{7,5}\Big(5^{4\alpha+2} \cdot 7n + \frac{301 \cdot 5^{4\alpha+1} - 5}{12}\Big) \equiv 0 \pmod 7, \tag{5}$$

$$B_{7,5}\Big(5^{4\alpha+2} \cdot 7n + \frac{77 \cdot 5^{4\alpha+2} - 5}{12}\Big) \equiv 0 \pmod 7, \tag{6}$$

$$B_{7,5}\Big(5^{4\alpha+2} \cdot 7n + \frac{469 \cdot 5^{4\alpha+1} - 5}{12}\Big) \equiv 0 \pmod 7, \tag{7}$$

$$B_{7,5}\Big(5^{4\alpha+4} n + \frac{5^{4\alpha+3} - 5}{12}\Big) \equiv 0 \pmod 7 \tag{8}$$

and

$$B_{7,5}\Big(5^{4\alpha+4} n + \frac{49 \cdot 5^{4\alpha+3} - 5}{12}\Big) \equiv 0 \pmod 7. \tag{9}$$

Theorem 1.4. *For all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$B_{7,25}\Big(5^{4\alpha+5}n + 5\cdot\frac{5^{4\alpha+4}-1}{4}\Big) \equiv g(4\alpha+2)B_{7,25}(5n) \pmod 7, \tag{10}$$

where

$$g(\alpha) = \Big(\frac{7\sqrt{13}}{26}+\frac{1}{2}\Big)\Big(\frac{\sqrt{13}}{2}+\frac{1}{2}\Big)^{\alpha} + \Big(\frac{-7\sqrt{13}}{26}+\frac{1}{2}\Big)\Big(\frac{-\sqrt{13}}{2}+\frac{1}{2}\Big)^{\alpha}. \tag{11}$$

Theorem 1.5. *Let $p_1, p_2, \cdots, p_r \geq 5$ be distinct primes with $p_i \equiv 3 \pmod 4$ for $1 \leq i \leq r$. Then for all integers $\beta_i, n, r \geq 0$ and for $p_r \nmid n$, we have*

$$B_{7,25}\Big(5np_r^{2\beta_r+1}\prod_{i=1}^{r-1}p_i^{2\beta_i+2} + 5\cdot\frac{\prod_{i=1}^{r}p_i^{2\beta_i+2}-1}{4}\Big) \equiv 0 \pmod 7.$$

Theorem 1.6. *For all integers $n, \alpha \geq 0$, $x \in \{13, 37\}$, $y \in \{301, 385, 469\}$ and $z \in \{1, 49\}$, we have*

$$B_{7,125}\Big(5^{4\alpha+6}n + \frac{x\cdot 5^{4\alpha+5}-65}{12}\Big) \equiv B_{7,5}\Big(5^{4\alpha+4}n + \frac{x\cdot 5^{4\alpha+3}-5}{12}\Big) \pmod 7, \tag{12}$$

$$B_{7,125}\Big(5^{4\alpha+6}\cdot 7n + \frac{y\cdot 5^{4\alpha+5}-65}{12}\Big) \equiv B_{7,5}\Big(5^{4\alpha+4}\cdot 7n + \frac{y\cdot 5^{4\alpha+3}-5}{12}\Big) \pmod 7 \tag{13}$$

and

$$B_{7,125}\Big(5^{4\alpha+8}n + \frac{z\cdot 5^{4\alpha+7}-65}{12}\Big) \equiv B_{7,5}\Big(5^{4\alpha+6}n + \frac{z\cdot 5^{4\alpha+5}-5}{12}\Big) \pmod 7. \tag{14}$$

Theorem 1.7. *For all integers $n, \alpha \geq 0$ and $(r,t) \in \{(43,3), (67,5)\}$, we have*

$$B_{11,5}\Big(5^{2\alpha+1}n + \frac{r\cdot 5^{2\alpha}-7}{12}\Big) \equiv h(\alpha)B_{11,5}(5n+t) \pmod{11}, \tag{15}$$

where

$$h(\alpha) = \Big(\frac{\sqrt{21}}{14}+\frac{1}{2}\Big)(6-\sqrt{21})^{\alpha} + \Big(\frac{-\sqrt{21}}{14}+\frac{1}{2}\Big)(6+\sqrt{21})^{\alpha}. \tag{16}$$

Theorem 1.8. *For all integers $n, \alpha, \beta \geq 0$ and $j \in \{43, 67\}$, we have*

$$B_{11,5}\Big(5^{12\beta+2\alpha+1}n + \frac{j\cdot 5^{12\beta+2\alpha}-7}{12}\Big) \equiv h(\alpha)B_{11,5}\Big(5^{12\beta+1}n + \frac{j\cdot 5^{12\beta}-7}{12}\Big) \pmod{11}, \tag{17}$$

where $h(\alpha)$ is as defined in (16).

From Theorems 1.4, 1.7 and using the fact that $B_{11,5}(58) \equiv B_{11,5}(95) \equiv 0 \pmod{11}$, $B_{7,25}(25) \equiv B_{7,25}(35) \equiv 0 \pmod 7$, we can obtain the following corollary:

Corollary 1.9. *For all integer $\alpha \geq 0$ and $(\beta, \delta) \in \{(11, 43), (18, 67)\}$, we have*

$$B_{11,5}\Big(5^{2\alpha+1}\cdot\beta + \frac{\delta\cdot 5^{2\alpha}-7}{12}\Big) \equiv 0 \pmod{11},$$

$$B_{7,25}\Big(5^{4\alpha+6} + 5\cdot\frac{5^{4\alpha+4}-1}{4}\Big) \equiv 0 \pmod 7$$

and

$$B_{7,25}\Big(5^{4\alpha+5}\cdot 7+5\cdot\frac{5^{4\alpha+4}-1}{4}\Big)\equiv 0 \pmod 7.$$

2. Preliminaries

The following 5-dissection formula for $(q;q)_\infty$ was first stated by Ramanujan [8, p. 212] without proof.

Lemma 2.1. *([8, p. 212]). We have*

$$(q;q)_\infty=(q^{25};q^{25})_\infty\Big(\frac{1}{R(q^5)}-q-q^2R(q^5)\Big) \tag{18}$$

where $R(q)=\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$.

Watson [12] presented a proof of (18) using the quintuple product identity.

Lemma 2.2. *([4, p. 165, eq. (7.4.14)]). We have*

$$\frac{1}{(q;q)_\infty}=\frac{(q^{25};q^{25})^5_\infty}{(q^5;q^5)^6_\infty}\Bigg(\frac{1}{R^4(q^5)}+\frac{q}{R^3(q^5)}+\frac{2q^2}{R^2(q^5)}+\frac{3q^3}{R(q^5)}+5q^4 \\ -3q^5R(q^5)+2q^6R^2(q^5)-q^7R^3(q^5)+q^8R^4(q^5)\Bigg). \tag{19}$$

Lemma 2.3. *([13]). We have*

$$\frac{1}{R^5(q)}-11q-q^2R^5(q)=\frac{(q;q)^6_\infty}{(q^5;q^5)^6_\infty}, \tag{20}$$

where $R(q)$ is as defined in Lemma 18.

The following lemmas will be important in our proofs.

Lemma 2.4. *If $\sum_{n=0}^{\infty}c(n)q^n=(q;q)^7_\infty$, then*

$$\sum_{n=0}^{\infty}c(5n+2)q^n=14(q;q)^6_\infty(q^5;q^5)_\infty+125q(q^5;q^5)^7_\infty.$$

Proof. From (18), we have

$$\sum_{n=0}^{\infty}c(n)q^n=(q^{25};q^{25})^7_\infty\Bigg(\frac{1}{R^7(q^5)}-\frac{7q}{R^6(q^5)}+\frac{14q^2}{R^5(q^5)}+\frac{7q^3}{R^4(q^5)}-\frac{49q^4}{R^3(q^5)}+\frac{14q^5}{R^2(q^5)} \\ +\frac{77q^6}{R(q^5)}-29q^7-77q^8R(q^5)+14q^9R^2(q^5)+49q^{10}R^3(q^5)+7q^{11}R^4(q^5) \\ -14q^{12}R^5(q^5)-7q^{13}R^6(q^5)-q^{14}R^7(q^5)\Bigg).$$

Extracting the terms involving q^{5n+2} in the above identity, dividing by q^2 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty}c(5n+2)q^n=(q^{25};q^{25})^7_\infty\Big(\frac{14}{R^5(q)}-29q-14q^2R^5(q)\Big) \\ =(q^{25};q^{25})^7_\infty\Big(14\Big(\frac{1}{R^5(q)}-11q-q^2R^5(q)\Big)+125q\Big). \tag{21}$$

Employing (20) in (21), we obtain (2.4). □

In [2], Adiga, Bayad and the present author employed (18) and (20) to prove the following lemmas:

Lemma 2.5. *If* $\sum_{n=0}^{\infty} a(n)q^n = (q;q)_\infty^9$, *then*

$$\sum_{n=0}^{\infty} a(5n+4)q^n = -90(q^5;q^5)_\infty^3(q;q)_\infty^6 - 625q(q^5;q^5)_\infty^9.$$

Lemma 2.6. *If* $\sum_{n=0}^{\infty} b(n)q^n = (q;q)_\infty^6$, *then*

$$\sum_{n=0}^{\infty} b(5n+1)q^n = -6(q;q)_\infty^6 - 25q(q^5;q^5)_\infty^6.$$

Lemma 2.7. *If* $\sum_{n=0}^{\infty} d(n)q^n = (q;q)_\infty^5$, *then*

$$\sum_{n=0}^{\infty} d(5n)q^n = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty}.$$

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we first prove the following lemma.

Lemma 3.1. *For all integers* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$B_{7,5}\Big(5^{4\alpha}n + 5\cdot\frac{5^{4\alpha}-1}{12}\Big) \equiv 5^{\alpha}B_{7,5}(n) \pmod 7. \tag{22}$$

Proof. Setting $k=7$, $s=5$ in (2) and then employing (1) with $p=7$, we deduce

$$\sum_{n=0}^{\infty} B_{7,5}(n)q^n = \frac{(q^7;q^7)_\infty(q^5;q^5)_\infty}{(q;q)_\infty^2} \equiv (q;q)_\infty^5(q^5;q^5)_\infty \pmod 7. \tag{23}$$

Extracting the terms of the form q^{5n} in (23), replacing q^5 by q and then substituting the Lemma 2.7, we obtain

$$\sum_{n=0}^{\infty} B_{7,5}(5n)q^n \equiv (q;q)_\infty \sum_{n=0}^{\infty} d(5n)q^n \equiv \frac{(q;q)_\infty^7}{(q^5;q^5)_\infty} \pmod 7. \tag{24}$$

In view of (1) with $p=7$ and (24), we have

$$\sum_{n=0}^{\infty} B_{7,5}(5n)q^n \equiv \frac{(q^7;q^7)_\infty}{(q^5;q^5)_\infty} \pmod 7. \tag{25}$$

Using (18), we can rewrite (25) as

$$\sum_{n=0}^{\infty} B_{7,5}(5n)q^n \equiv \frac{(q^7;q^7)_\infty}{(q^5;q^5)_\infty} = \frac{(q^{175};q^{175})_\infty}{R(q^{35})(q^5;q^5)_\infty} + 6\frac{q^7(q^{175};q^{175})_\infty}{(q^5;q^5)_\infty} + 6\frac{q^{14}R(q^{35})(q^{175};q^{175})_\infty}{(q^5;q^5)_\infty} \pmod 7. \tag{26}$$

Extracting the terms involving q^n for $n \equiv 2 \pmod 5$ in (26), dividing by q^2 and then replacing q^5 by q, we deduce

$$\sum_{n=0}^{\infty} B_{7,5}(25n+10)q^n \equiv 6q\frac{(q^{35};q^{35})_\infty}{(q;q)_\infty} \pmod 7.$$

Substituting (19) in the above congruence and then extracting those terms whose power of q is a multiple of 5, replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} B_{7,5}(125n+10)q^n \equiv 2q\frac{(q^7;q^7)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^6} \pmod 7. \tag{27}$$

By (18), (27) and (1) with $p=7$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{7,5}(125n+10)q^n \equiv& \frac{2q(q^{25};q^{25})_\infty (q^5;q^5)_\infty^5}{R(q^5)} + 5q^2(q^{25};q^{25})_\infty (q^5;q^5)_\infty^5 \\ &+ 5q^3R(q^5)(q^{25};q^{25})_\infty (q^5;q^5)_\infty^5 \pmod 7.\end{aligned} \tag{28}$$

Extracting those terms whose power of q is congruent to 2 modulo 5, dividing by q^2 and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} B_{7,5}(625n+260)q^n \equiv 5(q;q)_\infty^5(q^5;q^5)_\infty \equiv 5\sum_{n=0}^{\infty} B_{7,5}(n)q^n \pmod 7,$$

which gives that

$$B_{7,5}(625n+260) \equiv 5B_{7,5}(n) \pmod 7.$$

Congruence (22) follows from above congruence and by induction on $\alpha \geq 0$. This completes the proof of Lemma 3.1. □

Proof of Theorem 1.3. Equating the coefficients of q^{5n+1} and q^{5n+3} in (26), we obtain

$$B_{7,5}(25n+5) \equiv 0 \pmod 7, \tag{29}$$

$$B_{7,5}(25n+15) \equiv 0 \pmod 7. \tag{30}$$

If $p(n)$ denote the number of partitions of n, then Ramanujan [9] proved that, for all integers $n \geq 0$

$$p(7n+5) \equiv 0 \pmod 7. \tag{31}$$

By (26), we have

$$\sum_{n=0}^{\infty} B_{7,5}(25n)q^n \equiv \frac{(q^{35};q^{35})_\infty}{R(q^7)(q;q)_\infty} = \frac{(q^{35};q^{35})_\infty}{R(q^7)}\sum_{n=0}^{\infty} p(n)q^n \pmod 7.$$

Equating the coefficients of q^{7n+5} in the above congruence, we see that

$$\sum_{n=0}^{\infty} B_{7,5}(175n+125)q^n \equiv \frac{(q^5;q^5)_\infty}{R(q)}\sum_{n=0}^{\infty} p(7n+5)q^n \pmod 7,$$

From (31), we find that

$$B_{7,5}(175n+125) \equiv 0 \pmod 7. \tag{32}$$

In a similar way, from (26) and (31), we obtain

$$B_{7,5}(175n+160) \equiv 0 \pmod 7, \tag{33}$$

$$B_{7,5}(175n+195) \equiv 0 \pmod 7. \tag{34}$$

Equating the coefficients of q^{5n} and q^{5n+4} in (28), we see that

$$B_{7,5}(625n+10) \equiv 0 \pmod 7, \tag{35}$$

$$B_{7,5}(625n+510) \equiv 0 \pmod 7. \tag{36}$$

The congruences (3)-(9) follows from (29), (30), (32)-(36) and the Lemma 3.1. This completes the proof of Theorem 1.3.

4. Proofs of Theorems 1.4 and 1.5

In this section, we present proofs of Theorems 1.4 and 1.5. We first prove the following lemma.

Lemma 4.1. *For all integers $n \geq 0$ and $\alpha \geq 0$, we have*

(37)

$$B_{7,25}\Big(5^{\alpha+3}n + 5\cdot\frac{5^{\alpha+2}-1}{4}\Big) \equiv h(\alpha)B_{7,25}(25n+5) + g(\alpha)B_{7,25}(5n) \pmod 7,$$

where $g(\alpha)$ is as defined in (11) and $h(\alpha)$ is defined by

$$h(\alpha) = \Big(\frac{3\sqrt{13}}{26}+\frac{3}{2}\Big)\Big(\frac{\sqrt{13}}{2}+\frac{1}{2}\Big)^{\alpha} + \Big(\frac{-3\sqrt{13}}{26}+\frac{3}{2}\Big)\Big(\frac{-\sqrt{13}}{2}+\frac{1}{2}\Big)^{\alpha}.$$

Proof. From (2) and (1) with $p=7$, we have

$$\sum_{n=0}^{\infty} B_{7,25}(n)q^n = \frac{(q^7;q^7)_\infty (q^{25};q^{25})_\infty}{(q;q)_\infty^2} \equiv (q;q)_\infty^5 (q^{25};q^{25})_\infty \pmod 7. \tag{38}$$

Extracting the terms involving q^{5n} in (38), replacing q^5 by q and then substituting the Lemma 2.7, we obtain

$$\sum_{n=0}^{\infty} B_{7,25}(5n)q^n \equiv (q;q)_\infty^6 \pmod 7. \tag{39}$$

By (39) and Lemma 2.6, we deduce

$$\begin{aligned} \sum_{n=0}^{\infty} B_{7,25}(25n+5)q^n &\equiv \sum_{n=0}^{\infty} b(5n+1)q^n \\ &\equiv (q;q)_\infty^6 + 3q(q^5;q^5)_\infty^6 \pmod 7. \end{aligned} \tag{40}$$

In view of (39) and (40), we see that

$$B_{7,25}(125n+30) \equiv B_{7,25}(25n+5) + 3B_{7,25}(5n) \pmod 7, \tag{41}$$

which is same as (37) with $\alpha = 0$. Suppose (37) holds for $1 \leq \alpha \leq k$. Thus

(42)

$$B_{7,25}\Big(5^{k+3}n + 5\cdot\frac{5^{k+2}-1}{4}\Big) \equiv h(k)B_{7,25}(25n+5) + g(k)B_{7,25}(5n) \pmod 7$$

and
(43)
$$B_{7,25}\Big(5^{k+2}n+5\cdot\frac{5^{k+1}-1}{4}\Big)\equiv h(k-1)B_{7,25}(25n+5)+g(k-1)B_{7,25}(5n)\pmod 7.$$
It is a routine to verify that $h(k)$ and $g(k)$ satisfies the following recurrence relations:
$$h(k+1)=h(k)+3h(k-1) \tag{44}$$
and
$$g(k+1)=g(k)+3g(k-1). \tag{45}$$
Replacing n by $5^{k+1}n+\frac{5^{k+1}-1}{4}$ in (41) and then invoking (42)-(45), we see that
$$\begin{aligned}
&B_{7,25}\Big(5^{k+4}n+5\cdot\frac{5^{k+3}-1}{4}\Big)\\
&\equiv B_{7,25}\Big(5^{k+3}n+5\cdot\frac{5^{k+2}-1}{4}\Big)+3B_{7,25}\Big(5^{k+2}n+5\cdot\frac{5^{k+1}-1}{4}\Big)\\
&\equiv h(k)B_{7,25}(25n+5)+g(k)B_{7,25}(5n)+3\Big(h(k-1)B_{7,25}(25n+5)+g(k-1)B_{7,25}(5n)\Big)\\
&\equiv (h(k)+3h(k-1))B_{7,25}(25n+5)+(g(k)+3g(k-1))B_{7,25}(5n)\\
&=h(k+1)B_{7,25}(25n+5)+g(k+1)B_{7,25}(5n)\pmod 7.
\end{aligned}$$
That is, (37) holds for $\alpha=k+1$. This completes the proof of Lemma 37. □

Lemma 4.2. *For all integer $\alpha\geq 0$, we have*
$$h(4\alpha+2)\equiv 0\pmod 7. \tag{46}$$

Proof. The congruence (46) holds trivially for $\alpha=0$. Suppose that (46) is true for some $\alpha=k\geq 1$, i.e.,
$$h(4k+2)\equiv 0\pmod 7. \tag{47}$$
In view of (44), we have
$$\begin{aligned}
h(4k+6)=&h(4k+5)+3h(4k+4)\\
=&4h(4k+4)+3h(4k+3)\\
=&7h(4k+3)+15h(4k+2).
\end{aligned}$$
By (47), we see that
$$h(4k+6)\equiv 0\pmod 7.$$
Hence the proof. □

Proof of Theorem 1.4. Replacing α by $4\alpha+2$ in (37) and then using (46), we obtain the congruence (10). This completes the proof of Theorem 1.4.

Proof of Theorem 1.5. From [4, Theorem 1.3.9], we recall the Jacobi's identity,
$$(q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{48}$$

From (39) and (48), we have

$$\sum_{n=0}^{\infty} B_{7,25}(5n)q^{8n+2} \equiv \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^m(2m+1)(2k+1)q^{(2m+1)^2+(2k+1)^2} \pmod 7.$$

Note that if $8n+2 \neq (2k+1)^2+(2m+1)^2$ for $n,k,m \geq 0$, then $B_{7,25}(5n) \equiv 0 \pmod 7$. Let $p_1, p_2, \cdots, p_r \geq 5$ be distinct primes with $p_i \equiv 3 \pmod 4$ for $1 \leq i \leq r$. Let $v_{p_i}(N)$ denote the power of prime p_i in the unique prime factorisation of N. If $N = x^2+y^2$, then $v_{p_i}(N)$ must be even because $p_i \equiv 3 \pmod 4$. Hence, if $p_r \nmid n$, then

$$v_{p_r}\left(8\left(np_r^{2\beta_r+1}\prod_{i=1}^{r-1} p_i^{2\beta_i+2} + \frac{\prod_{i=1}^{r} p_i^{2\beta_i+2}-1}{4}\right)+2\right) = 2\beta_r+1,$$

and so, for $p_r \nmid n$, we have

$$B_{7,25}\left(5np_r^{2\beta_r+1}\prod_{i=1}^{r-1} p_i^{2\beta_i+2} + 5\frac{\prod_{i=1}^{r} p_i^{2\beta_i+2}-1}{4}\right) \equiv 0 \pmod 7.$$

This completes the proof of Theorem 1.5.

5. Proof of Theorem 1.6

Before proving the Theorem 1.6, we first prove the following theorem concerning $B_{7,5}(n)$ and $B_{7,125}(n)$.

Theorem 5.1. *For all integer $n \geq 0$, we have*

$$B_{7,125}(175n+155) \equiv B_{7,5}(7n+6) \pmod 7, \tag{49}$$

$$B_{7,125}(625n+5) \equiv B_{7,5}(25n) \pmod 7, \tag{50}$$

$$B_{7,125}(625n+130) \equiv B_{7,5}(25n+20) \pmod 7 \tag{51}$$

and

$$B_{7,125}(625n+255) \equiv B_{7,5}(25n+10)+4B_{7,5}(n) \pmod 7. \tag{52}$$

Proof. By (2), (1) with $p=7$ and the Lemma 2.7, we obtain

$$\sum_{n=0}^{\infty} B_{7,125}(5n)q^n \equiv \frac{(q^{25};q^{25})_\infty (q;q)_\infty^6}{(q^5;q^5)_\infty} \pmod 7. \tag{53}$$

In view of Lemma 2.6, (23), (53) and (1), we see that

(54)

$$\begin{aligned}\sum_{n=0}^{\infty} B_{7,125}(25n+5)q^n &\equiv \sum_{n=0}^{\infty} B_{7,5}(n)q^n + 2q\frac{(q^5;q^5)_\infty^7}{(q;q)_\infty} \\ &\equiv \sum_{n=0}^{\infty} B_{7,5}(n)q^n + 2(q^{35};q^{35})_\infty \sum_{n=0}^{\infty} p(n)q^{n+1} \pmod 7.\end{aligned}$$

Extracting those terms whose power of q is congruent to 6 modulo 7 in (54), we find that

(55)

$$\sum_{n=0}^{\infty} B_{7,125}(175n+155)q^n \equiv \sum_{n=0}^{\infty} B_{7,5}(7n+6)q^n + 2(q^5;q^5)_\infty \sum_{n=0}^{\infty} p(7n+5)q^n \pmod 7.$$

Congruence (49) follows from (31) and (55).
Applying (19) into (54), we deduce
(56)
$$\sum_{n=0}^{\infty} B_{7,125}(125n+5)q^n \equiv \sum_{n=0}^{\infty} B_{7,5}(5n)q^n + 3q(q^5;q^5)_\infty^5(q;q)_\infty \pmod 7.$$
Employing (18) into (56), equating the coefficients of q^{5n} and q^{5n+4}, we arrive to (50) and (51), respectively. From (18) and (56), we have
(57)
$$\sum_{n=0}^{\infty} B_{7,125}(625n+255)q^n \equiv \sum_{n=0}^{\infty} B_{7,5}(25n+10)q^n + 4(q;q)_\infty^5(q^5;q^5)_\infty \pmod 7.$$
Congruence (52) follows from (57) and (23). This completes the proof. □

Proof of Theorem 1.6. Replacing n by $5^{4\alpha+2}n + \frac{x\cdot 5^{4\alpha+1}-5}{12}$ where $x \in \{13, 37\}$ in (52) and then employing (29) and (30), we arrive at (12). In a similar way, (13) follows from (52), (32)-(34) and (14) follows from (52), (35), (36). Hence the proof.

6. Proofs of Theorems 1.7 and 1.8

In this section, we present proofs of Theorems 1.7 and 1.8. We first prove the following lemma.

Lemma 6.1. *Let $\sum_{n=0}^{\infty} C(n)q^n = (q;q)_\infty^3(q^5;q^5)_\infty^7$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*
(58)
$$B_{11,5}\Big(5^{2\alpha}n + \frac{19\cdot 5^{2\alpha}-7}{12}\Big) \equiv h(\alpha)B_{11,5}(n+1) + \frac{h(\alpha+1)-3h(\alpha)}{4}C(n) \pmod{11},$$
where $h(\alpha)$ is as defined in (16).

Proof. Setting $k = 11$ and $s = 5$ in (2) and then employing (1) with $p = 11$, we see that
$$(59)\quad \sum_{n=0}^{\infty} B_{11,5}(n)q^n = \frac{(q^{11};q^{11})_\infty(q^5;q^5)_\infty}{(q;q)_\infty^2} \equiv (q;q)_\infty^9(q^5;q^5)_\infty \pmod{11}.$$
Extracting the terms of the form q^{5n+4} in (59), dividing by q^4, replacing q^5 by q and then substituting the Lemma 2.5, we obtain
(60)
$$\begin{aligned}\sum_{n=0}^{\infty} B_{11,5}(5n+4)q^n &\equiv (q;q)_\infty \sum_{n=0}^{\infty} a(5n+4)q^n \\ &\equiv 9(q;q)_\infty^7(q^5;q^5)_\infty^3 + 2q(q;q)_\infty(q^5;q^5)_\infty^9 \pmod{11}.\end{aligned}$$
In view of (18), we can rewrite (60) as
$$\sum_{n=0}^{\infty} B_{11,5}(5n+4)q^n$$
$$\equiv 9(q;q)_\infty^7(q^5;q^5)_\infty^3 + (q^{25};q^{25})_\infty(q^5;q^5)_\infty^9\left(\frac{2q}{R(q^5)} + 9q^2 + 9q^3R(q^5)\right) \pmod{11},$$

which yields

$$\begin{aligned}\sum_{n=0}^{\infty} B_{11,5}(25n+14)q^n \equiv& 9(q;q)_\infty^3 \sum_{n=0}^{\infty} c(5n+2)q^n + 9(q^5;q^5)_\infty (q;q)_\infty^9 \\ \equiv& 3(q^5;q^5)_\infty (q;q)_\infty^9 + 3q(q^5;q^5)_\infty^7 (q;q)_\infty^3 \pmod{11}.\end{aligned}$$

From (59) and the above congruence, we obtain

$$B_{11,5}(25n+39) \equiv 3B_{11,5}(n+1) + 3C(n) \pmod{11}, \tag{61}$$

where $C(n)$ is defined by

$$\sum_{n=0}^{\infty} C(n)q^n = (q^5;q^5)_\infty^7 (q;q)_\infty^3. \tag{62}$$

Employing (18) in (62), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} C(n)q^n =& (q^{25};q^{25})_\infty^3 (q^5;q^5)_\infty^7 \\ & \times \left(\frac{1}{R^3(q^5)} - \frac{3q}{R^2(q^5)} + 5q^3 - 3q^5R^2(q^5) - q^6R^3(q^5) \right),\end{aligned} \tag{63}$$

which yields

$$\sum_{n=0}^{\infty} C(5n+3)q^n = 5(q^5;q^5)_\infty^3 (q;q)_\infty^7.$$

Extracting the terms involving q^{5n+2}, dividing by q^2, replacing q^5 by q and then using the Lemma 2.4, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} C(5n+3)q^n =& 5(q;q)_\infty^3 \sum_{n=0}^{\infty} c(5n+2)q^n \\ \equiv& 4(q;q)_\infty^9 (q^5;q^5)_\infty + 9q(q^5;q^5)_\infty^3 (q;q)_\infty^7 \pmod{11}.\end{aligned}$$

In view of (59), (67) and then above congruence, we deduce

$$C(25n+38) \equiv 4B_{11,5}(n+1) + 9C(n) \pmod{11}. \tag{64}$$

Now, we will prove (58) by induction on $\alpha \geq 0$. It is trivial to verify that (58) is true for $\alpha = 0$. Suppose (58) holds for some $\alpha \geq 1$. It is easy to check that $h(\alpha)$ satisfies the following relation:

$$h(\alpha+1) = 12h(\alpha+1) - 15h(\alpha). \tag{65}$$

Replacing n by $25n+38$ in (58), employing (64), (65) and (61) in the resulting congruence, we obtain

$$\begin{aligned}
&B_{11,5}\Big(5^{2\alpha+2}n+\frac{19\cdot 5^{2\alpha+2}-7}{12}\Big)\\
&\equiv h(\alpha)B_{11,5}(28n+39)+\frac{h(\alpha+1)-3h(\alpha)}{4}C(25n+38)\\
&\equiv h(\alpha)\Big(3B_{11,5}(n+1)+3C(n)\Big)+\frac{h(\alpha+1)-3h(\alpha)}{4}\Big(4B_{11,5}(n+1)+9C(n)\Big)\\
&=(h(\alpha+1))B_{11,5}(n+1)+\frac{9h(\alpha+1)-15h(\alpha)}{4}C(n)\\
&=h(\alpha+1)B_{11,5}(n+1)+\frac{h(\alpha+2)-3h(\alpha+1)}{4}C(n) \pmod{11}.
\end{aligned}$$

That is, (58) holds for $\alpha+1$. Hence the proof. □

Proof of Theorem 1.7. Equating the coefficients of q^{5n+2} and q^{5n+4} in (63), we obtain

$$C(5n+2)=0 \text{ and } C(5n+4)=0. \tag{66}$$

Congruence (15) follows from (58) and (66). This completes the proof of Theorem 1.7.

Proof of Theorem 1.8. In view of (61), (64) and by induction on $\beta\geq 0$, we can prove the following congruence relation:

(67)
$$C\Big(5^{2\beta}n+\frac{19\cdot 5^{2\beta}-19}{12}\Big)\equiv d(\beta)B_{11,5}(n+1)+\frac{d(\beta+1)-3d(\beta)}{4}C(n) \pmod{11},$$

where $d(\beta)$ is defined by

$$d(\beta)=\frac{2}{\sqrt{21}}\Big(\Big(6+\sqrt{21}\Big)^{\beta}-\Big(6-\sqrt{21}\Big)^{\beta}\Big).$$

It is trivial to check that

$$d(\beta+1)=12d(\beta)-15d(\beta-1). \tag{68}$$

Utilizing (68), we can prove the following congruence by induction on $\beta\geq 0$

$$d(6\beta)\equiv 0 \pmod{11}. \tag{69}$$

First, replacing β by 6β in (67), using (69) and then replacing n by $5n+2$ and $5n+4$, respectively, we obtain

$$C\Big(5^{12\beta+1}n+\frac{j\cdot 5^{12\beta}-19}{12}\Big)\equiv 0 \pmod{11}, \tag{70}$$

where $j\in\{43,67\}$. Congruence (17) follows from (58) and (70). This completes the proof of Theorem 1.8.

References

[1] C. Adiga and D. Ranganatha, *A simple proof of a conjecture of Dou on (3, 7)-regular bipartitions modulo 3*, Integers 17 (2017), #A14.

[2] C. Adiga, A. Bayad and D. Ranganatha, *Some new congruences for (s,t)-regular bipartition functions*, Publ. Inst. Math. (2017) (to appear).

[3] C. Adiga and D. Ranganatha, *Congruences for 7 and 49-regular partitions modulo powers of 7*, Ramanujan J. (2017), https://doi.org/10.1007/s11139-017-9936-z.

[4] B.C. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer. Math. Soc., Providence, RI, 2006.

[5] D.Q.J. Dou, *Congruences for (3, 11)-regular bipartitions modulo 11*, Ramanujan J. 40 (3) (2016), 535–540.

[6] B.L.S. Lin, *Arithmetic of the 7-regular bipartition function modulo 3*, Ramanujan J. 37 (2015), 469–478.

[7] B.L.S. Lin, *An infinite family of congruences modulo 3 for 13-regular bipartitions*, Ramanujan J. 39 (2016), 169–178.

[8] S. Ramanujan, Collected Papers, Cambridge University Press, Cambridge, 1927.

[9] S. Ramanujan, *Some propeties of $p(n)$, the number of partitions of n*, Proc. Cambridge Philos. Soc. 19 (1919), 207–210.

[10] D. Ranganatha, *On a Ramanujan-type congruence for bipartitions with 5-cores*, J. Integer Sequences 19 (2016), Article 16.8.1. (2016).

[11] D. Ranganatha, *Ramanujan-type congruences modulo powers of 5 and 7*, Indian J. Pure Appl. Math. 48(3) (2017), 449–465

[12] G.N. Watson, *Ramanujan's Vermutung über Zerfällungsanzahlem*, J. reine angew. Math. 179 (1938), 97–128.

[13] G.N. Watson, *Theorems stated by Ramanujan (VII): theorems on contuined fractions*, J. London Math. Soc. 4 (1929), 39–48.

[14] L. Wang, *Congruences modulo powers of 5 for two restricted bipartitions*, Ramanujan J. 44(3) (2017), 471–491.

[15] E.X.W. Xia, *Congruences for some ℓ-regular partitions modulo ℓ*, J. Number Theory 152 (2015), 105–117.

[16] E.X.W. Xia and X.M. Yao, *Arithmetic properties for (s,t)-regular bipartition functions*, J. Number Theory 171 (2017), 1–17.

Department of Mathematics, School of Physical Sciences,, Central University of Karnataka,, Kalaburagi-585376, Karnataka, India,

Email address: `ddranganatha@gmail.com`